Some Remarks on Logic and Topology

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Reminder
Basic Definitions

- A topology is a set X (the <u>universe</u>) and a collection τ of subsets of X (the <u>open sets</u>) such that:
- τ contains X and $\emptyset$
- The union of any collection of elements of τ is in τ
- The intersection of any finite number of elements of τ is in τ

Reminder
Basic Definitions

- The complement of a subset A of X, which we will note by A' is defined as X \ A.
- The complement of an open se[t is] closed.

Some people write Ac for the complement of a set. In order to avoid confusion with the closure of a set, I will write A' for the complement and A- for the closure.

Reminder
Basic Definitions

- A set can be both closed and open (specifically, both X and $\emptyset$ are bo closed and open in any topology)

Some people like the word "clopen" for a set which is both closed and open. I don't. It is very widely used, though.

- a set can also be neither closed nor open.

Reminder (Very) Basic Properties

- $(A \cup B)' = A' \cap B'$
- $X \setminus (A \cup B) = X \setminus A \cap X \setminus B$
- $\forall x \; x \in X \wedge x \notin (A \cup B) \leftrightarrow$
  $x \in X \wedge x \notin A \wedge x \notin B$

Reminder
(Very) Basic Properties

- $(A \cup B)' = A' \cap B'$
- $X \setminus (A \cup B) = X \setminus A \cap X \setminus B$
- $\forall x\ x \in X \wedge \neg(x \in (A \cup B)) \leftrightarrow x \in X \wedge x \notin A \wedge x \notin B$

Reminder
(Very) Basic Properties

- $(A \cup B)' = A' \cap B'$
- $X \setminus (A \cup B) = X \setminus A \cap X \setminus B$
- $\forall x\ x \in X \wedge \neg(x \in A \vee x \in B)) \leftrightarrow$
  $x \in X \wedge x \notin A \wedge x \notin B$

Reminder (Very) Basic Properties

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Reminder
(Very) Basic Properties

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- $X \setminus (A \cap B) = X \setminus A \cup X \setminus B$
- $\forall x \; x \in X \wedge x \notin (A \cap B) \leftrightarrow$
  $x \in X \wedge (x \notin A \vee x \notin B)$

Reminder
Basic Definitions

- Given a set A, the interior A^o of A is the union of all open sets O such that O $\subseteq$ A.
- Given a set A, the closure A^- of A is the intersection of all closed sets C such that A $\subseteq$ C.
- Evidently, we have $A^o \subseteq A \subseteq A^-$

Reminder
Basic Definitions

- Evidently, we have $A^o \subseteq A \subseteq A^-$
- The boundary of A, δA is defined as $A^- \setminus A^o$

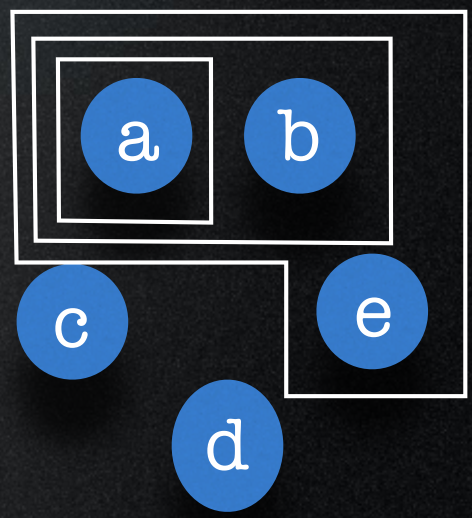
Reminder
Example 1

$X = \{a,b,c,d,e\}$

$\tau = \{\emptyset, X, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$

$\tau' = \{\emptyset, X, \{b,c,d,e\}, \{a,b,e\}, \{b,e\}, \{a\}\}$

$A = \{a,b\}$



$A^{o} = \{a\} \cup \emptyset = \{a\}$

$A^{-} = X \cap \{a,b,e\} = \{a,b,e\}$

$\delta A = \{a,b,e\} \setminus \{a\} = \{b,e\}$

Reminder Example 2

$X = \{a,b,c,d,e\}$

$\tau = \{\emptyset, X, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$

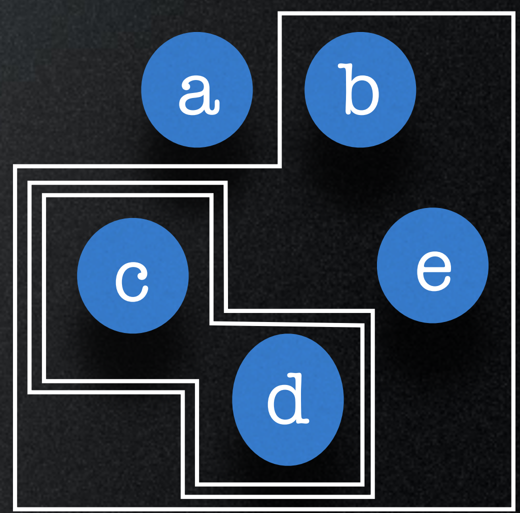
$\tau' = \{\emptyset, X, \{b,c,d,e\}, \{a,b,e\}, \{b,e\}, \{a\}\}$

$A = \{c,d\}$

$A^{o} = \{c,d\} \cup \emptyset = \{c,d\}$

$A^{-} = X \cap \{b,c,d,e\} = \{b,c,d,e\}$

$\delta A = \{b,c,d,e\} \setminus \{c,d\} = \{b,c\}$

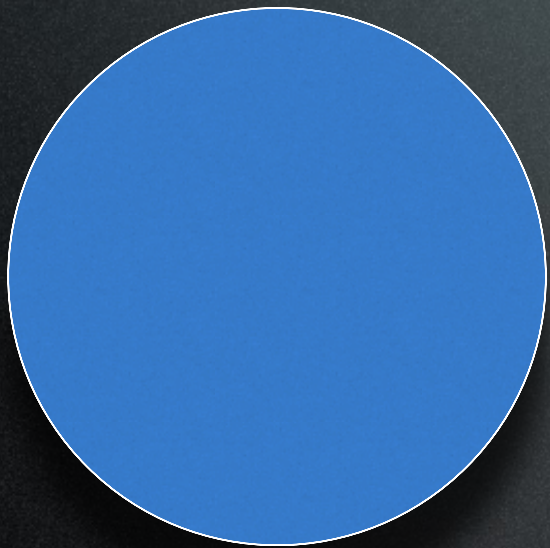


Reminder
Example 1

$X = \langle 0,0\rangle - \langle 5,5\rangle$

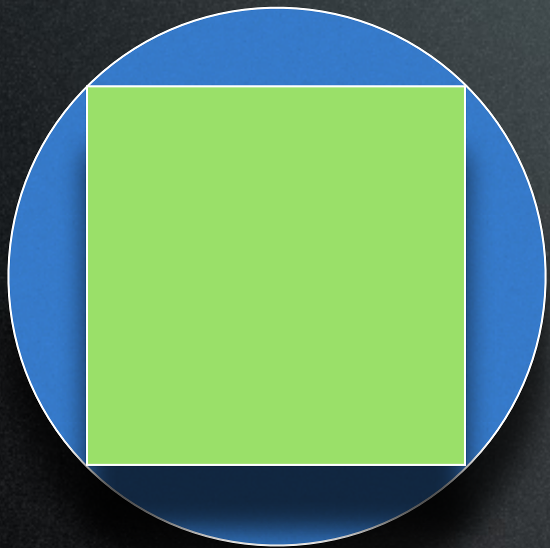
$\tau = \{\varnothing, \langle 0,5\rangle, <, \{b,c,d\}\}$

Reminder
Example 1



- Approximating a circle using rectangles (fortunately we have infinite unions)

Reminder Example 1



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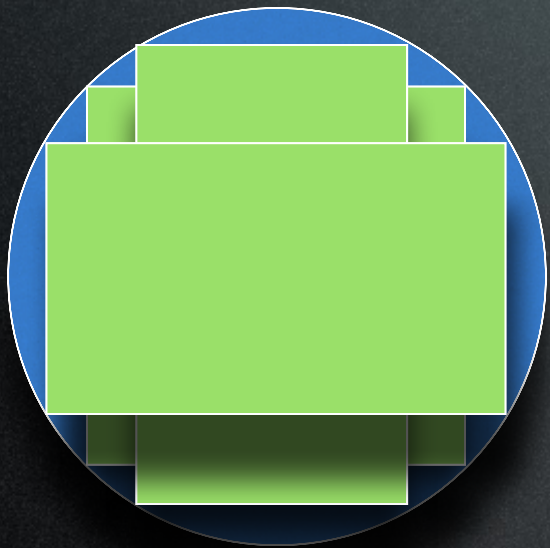

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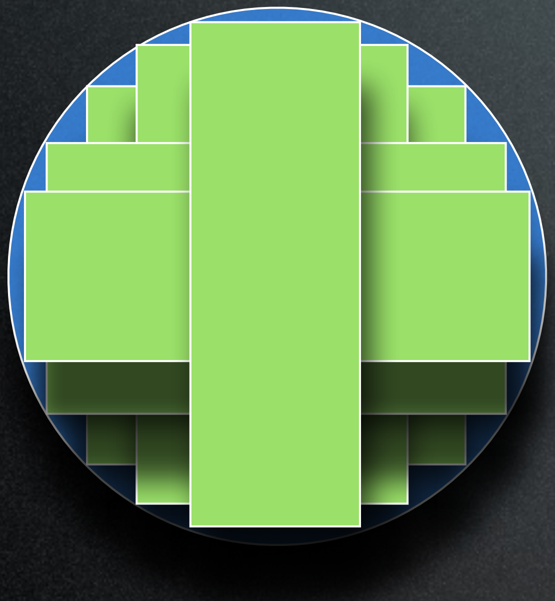
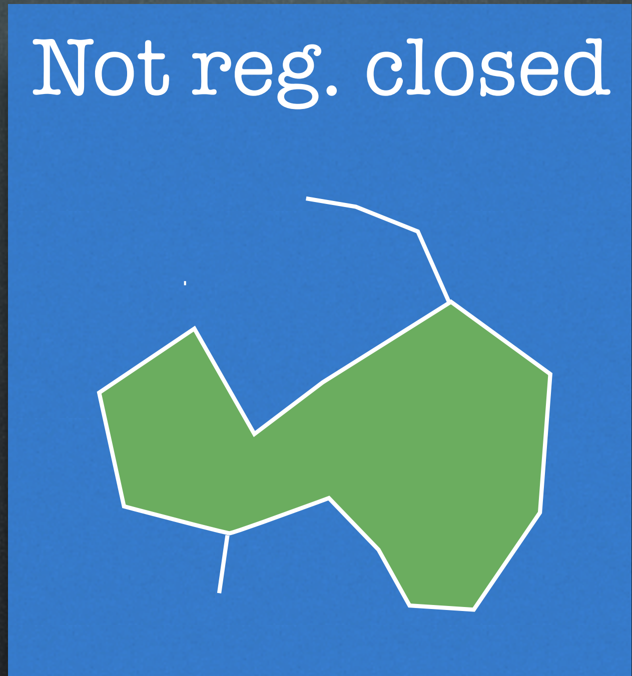


- Approximating a circle using rectangles (fortunately we have infinite unions)

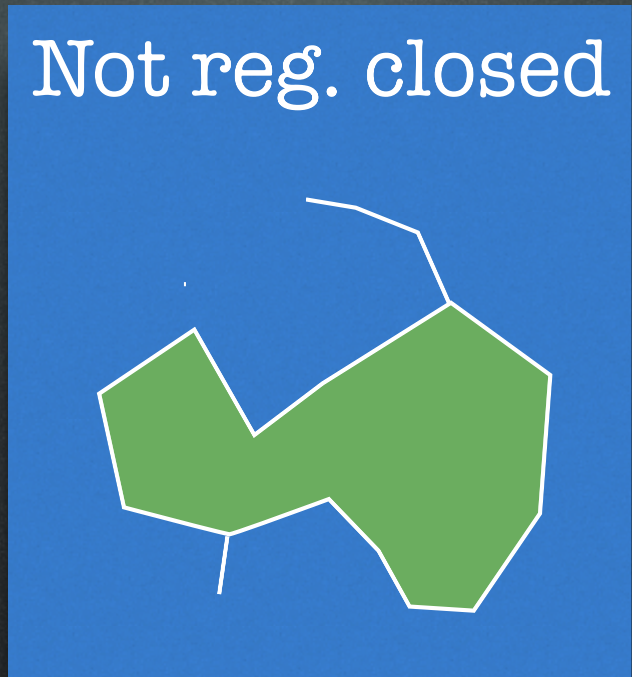
Regular Closed Sets



- A is <u>regular closed</u> iff $A = A^{\circ -}$
- roughly speaking, this means no “loose points” and no “hanging lines”

Regular Closed Sets

A



B

- A is regular closed iff $A = A^{\circ -}$
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Regular Closed Sets

A°

B°

- A is regular closed iff $A = A^{\circ -}$
- roughly speaking, this means no "loose points" and no "hanging lines"

Regular Closed Sets

A^{o-}

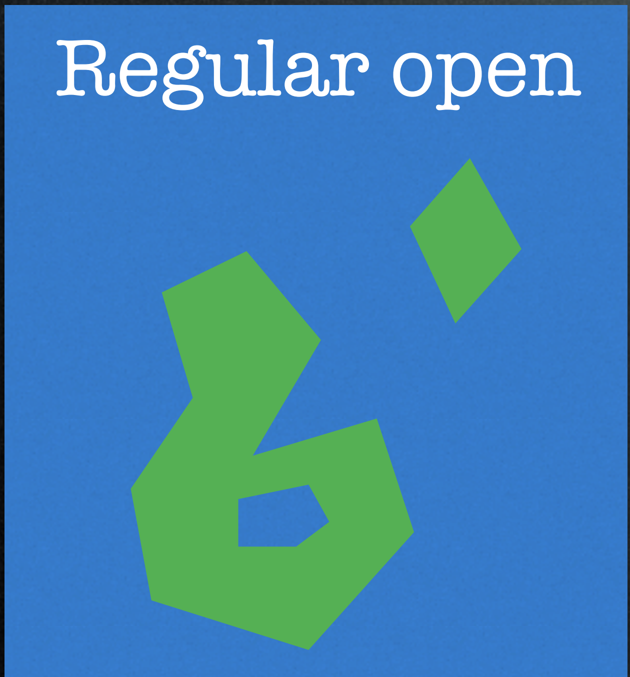
$=A$

B^{o-}

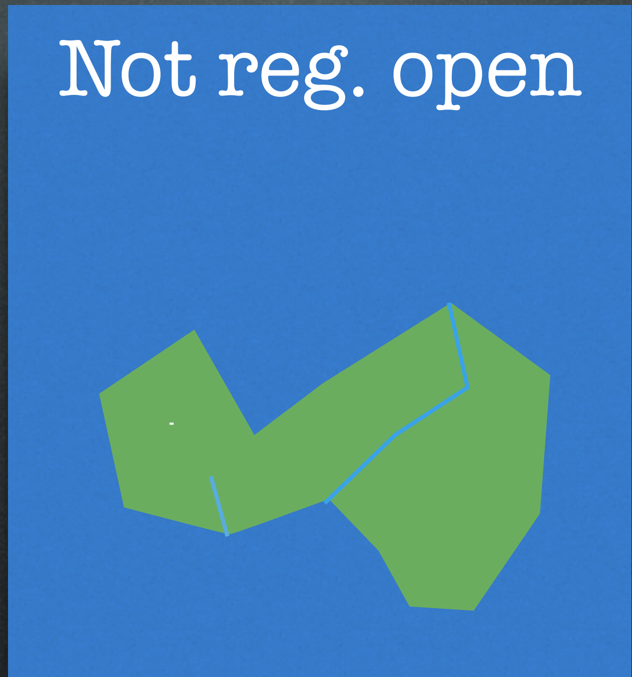
$\neq B$

- A is regular closed iff $A = A^{o-}$
- roughly speaking, this means no "loose points" and no "hanging lines"

Regular Open Sets



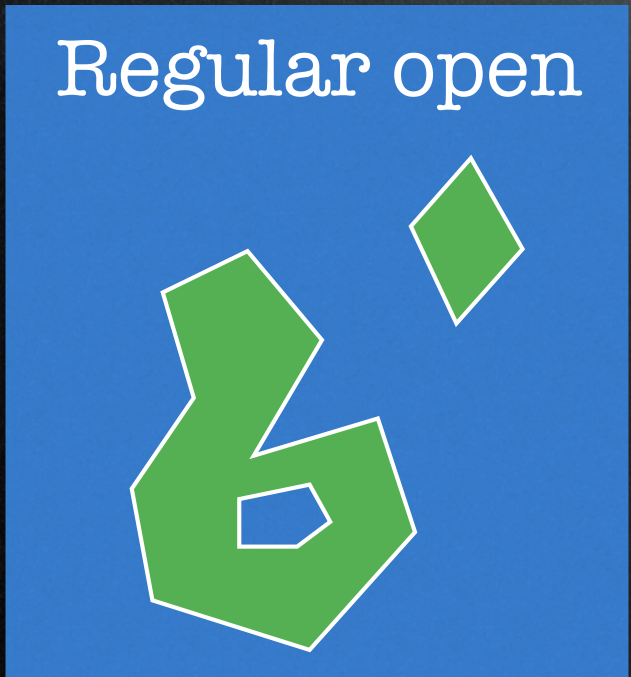
A



B

- A is <u>regular open</u> iff $A = A^{-o}$
- roughly speaking, this means no "holes" and no "cracks"

Regular Open Sets



A^-

B^-

- A is <u>regular open</u> iff $A = A^{-\circ}$
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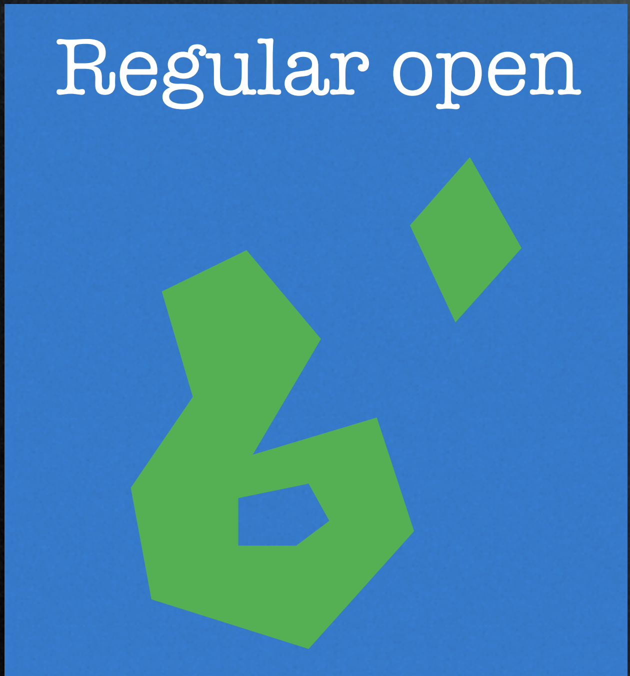
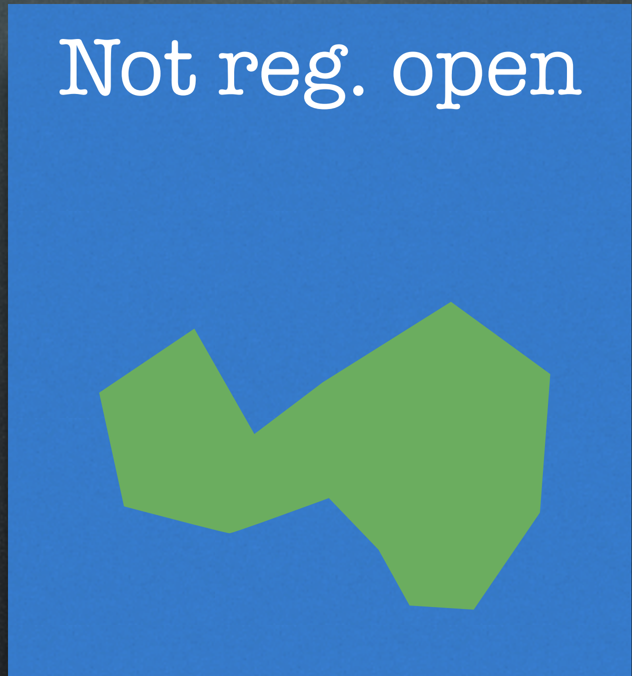
Regular Open Sets



A^{-o}

$= A$



B^{-o}

$\neq B$

- A is regular open iff $A = A^{-o}$
- roughly speaking, this means no "holes" and no "cracks"

Regular Sets

- Suppose A is a regular open set, that is $A = A^{-o}$.
- Thus we have, for its complement A' that $A' = A^{-o\prime} = A^{-\prime-} = A^{\prime o-}$
- In addition, we have $A^{-} = A^{-o-}$
- In other words, both A' and A regular closed.

Similar remarks hold for regular closed sets. Given a regular closed set A, both its complement and its interior are regular open.

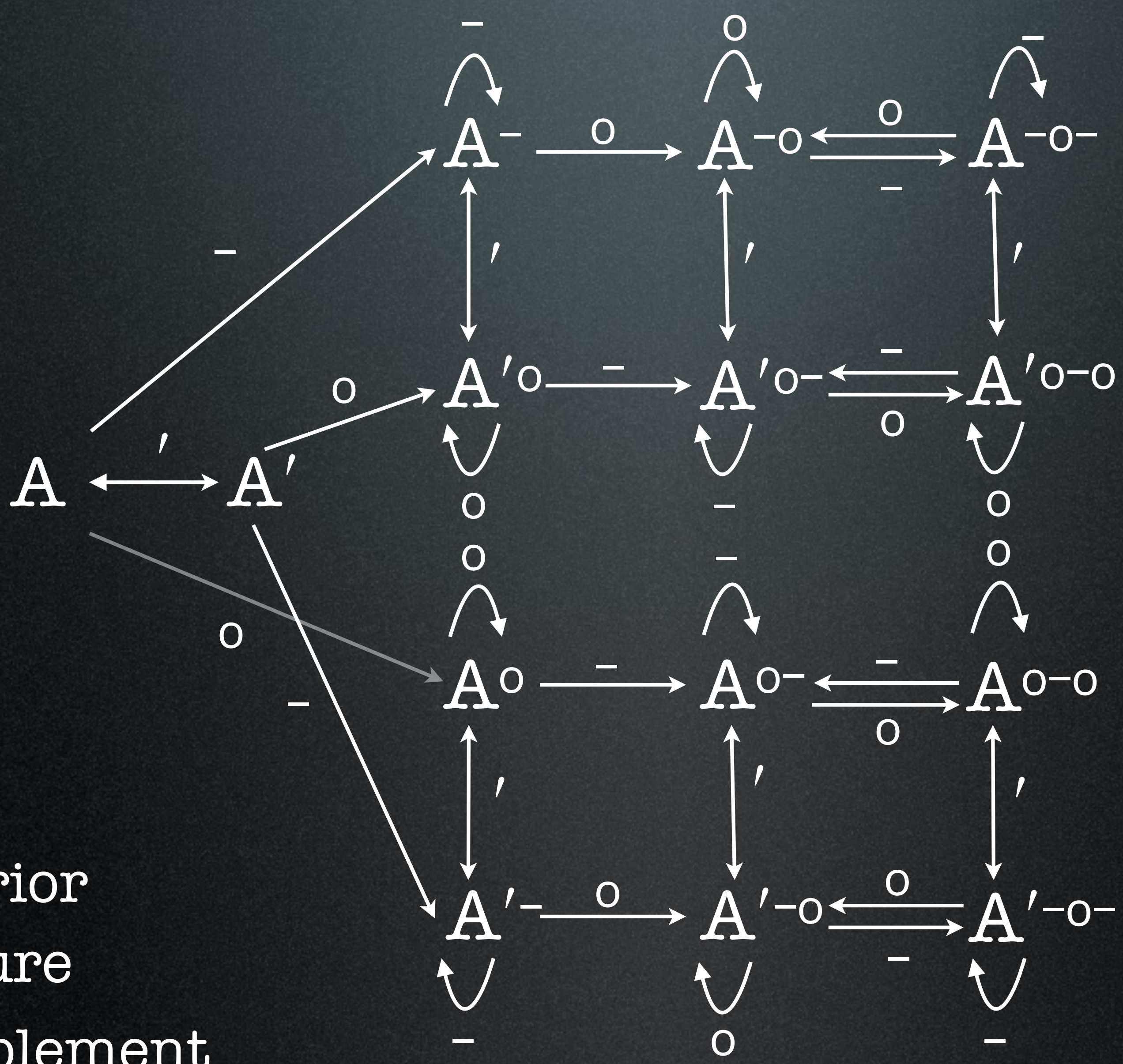
Combinations

- Given a region variable A, how many (potentially) different regions can we construct using the closure, interior and complement relations?
- Our possibilities are limited by the fact that a double complement is the identity function and that the closure and interior operations are idempotent.

Combinations

- Equivalences
  - $A'' = A$
  - $A^{oo} = A^{o}$
  - $A^{--} = A^{-}$
  - $A^{o-o-} = A^{o-}$
  - $A^{-o-o} = A^{-o}$
- Non-equivalent
  - A
  - A^{-}, A^{o}
  - A^{-o}, A^{o-}
  - A^{-o-}, A^{o-o}
- and their negations

Combinations

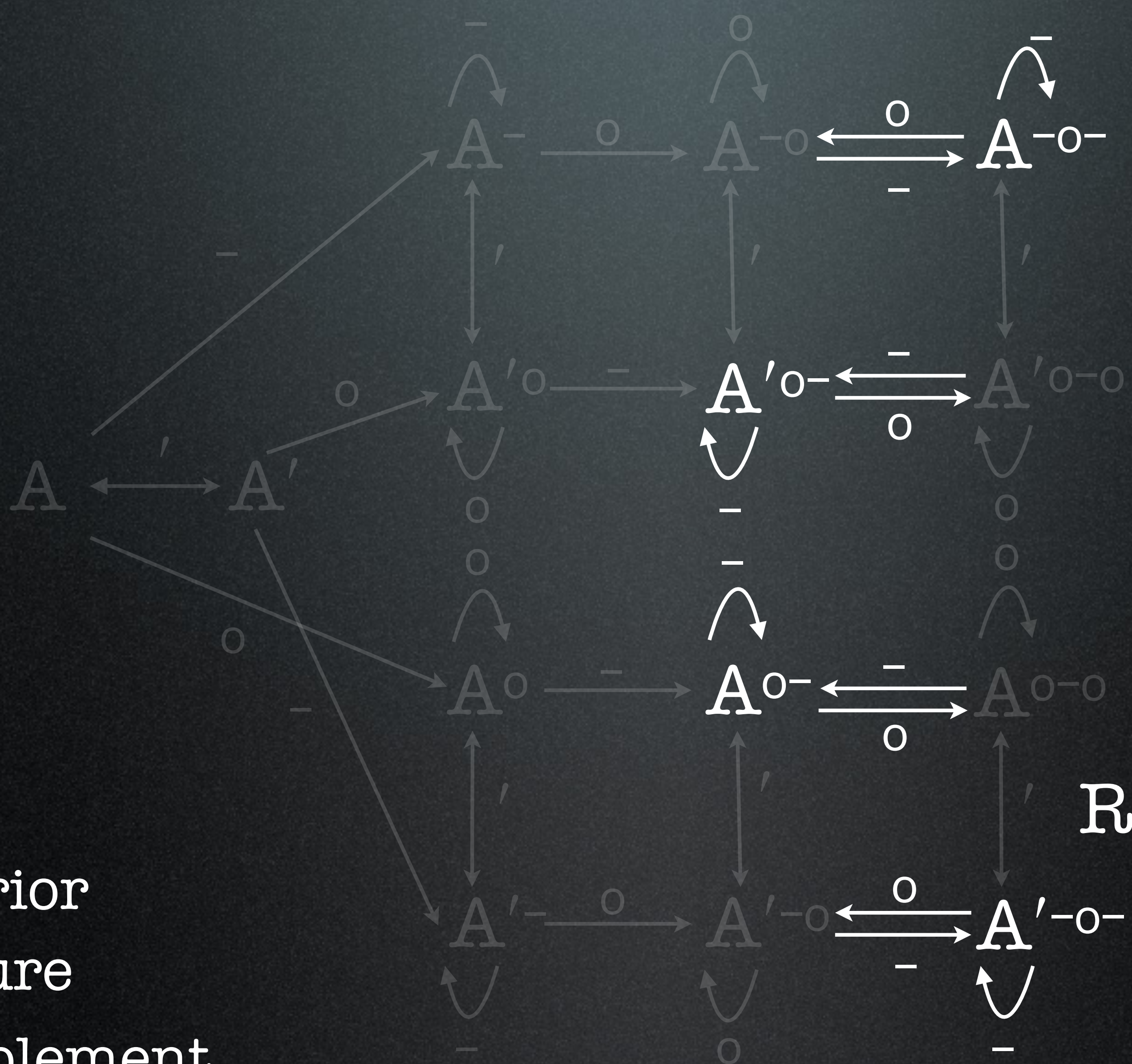


o : interior

– : closure

′ : complement

Combinations



Regular Closed

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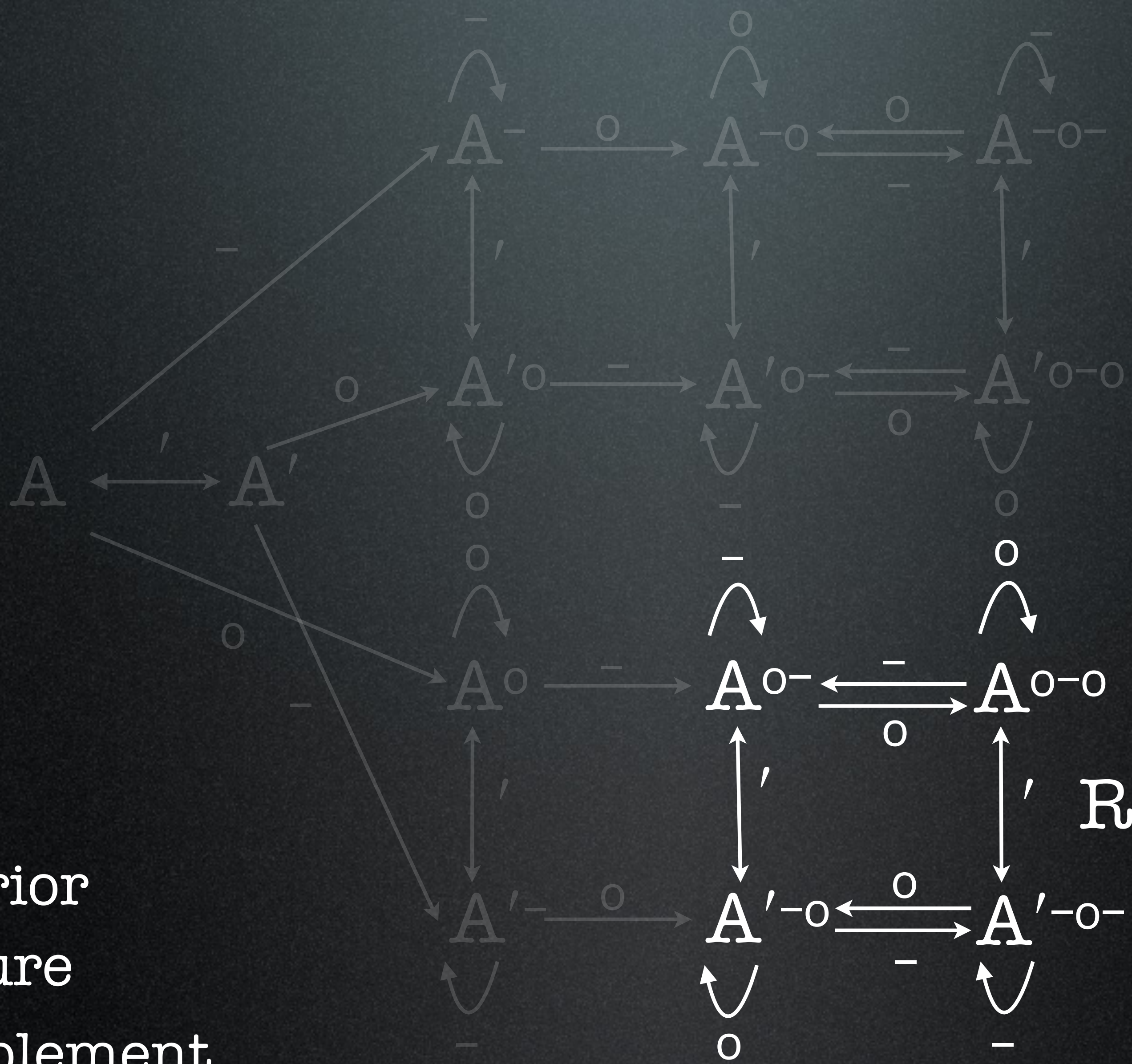
Combinations



Regular Closed

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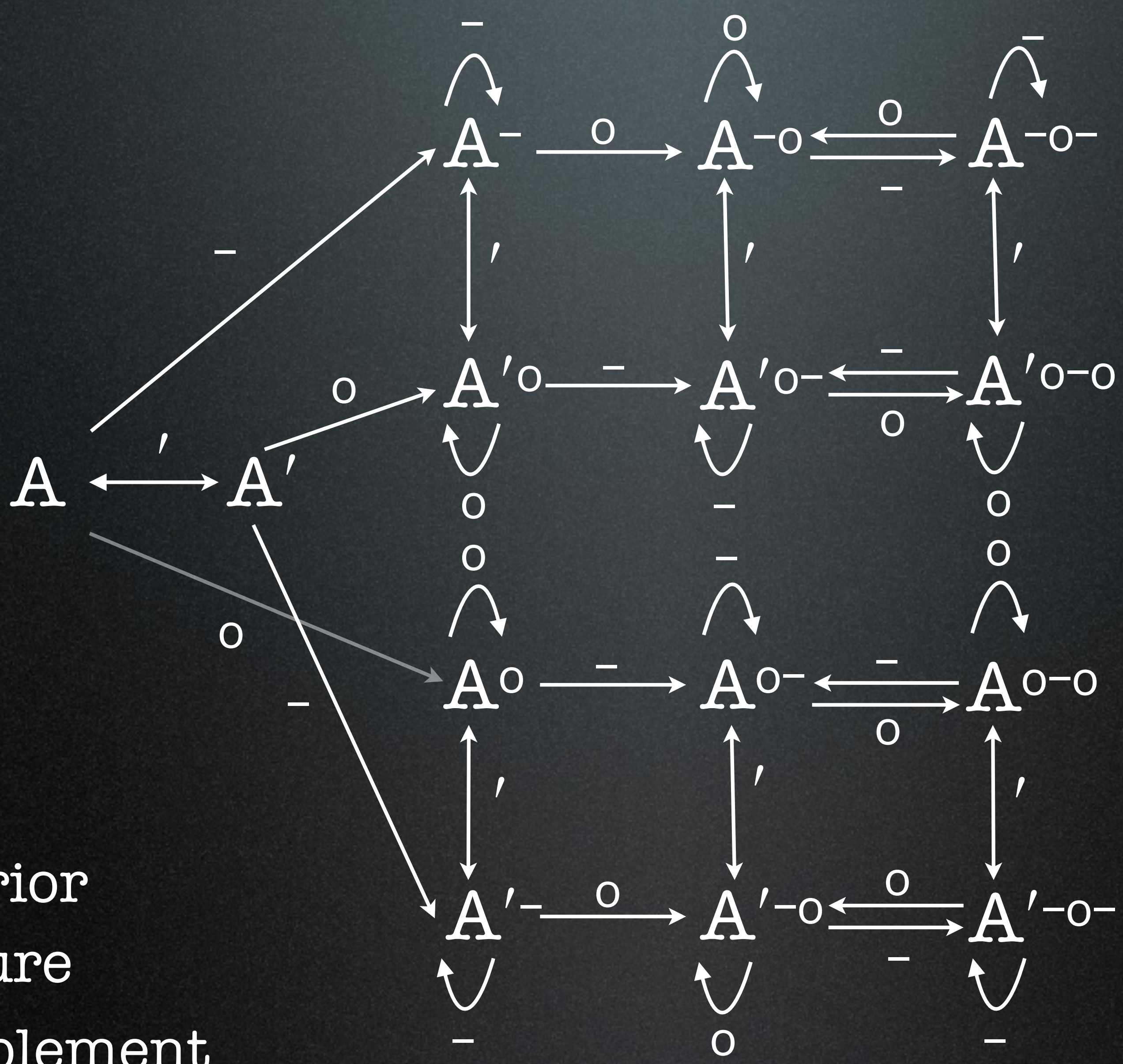
Combinations



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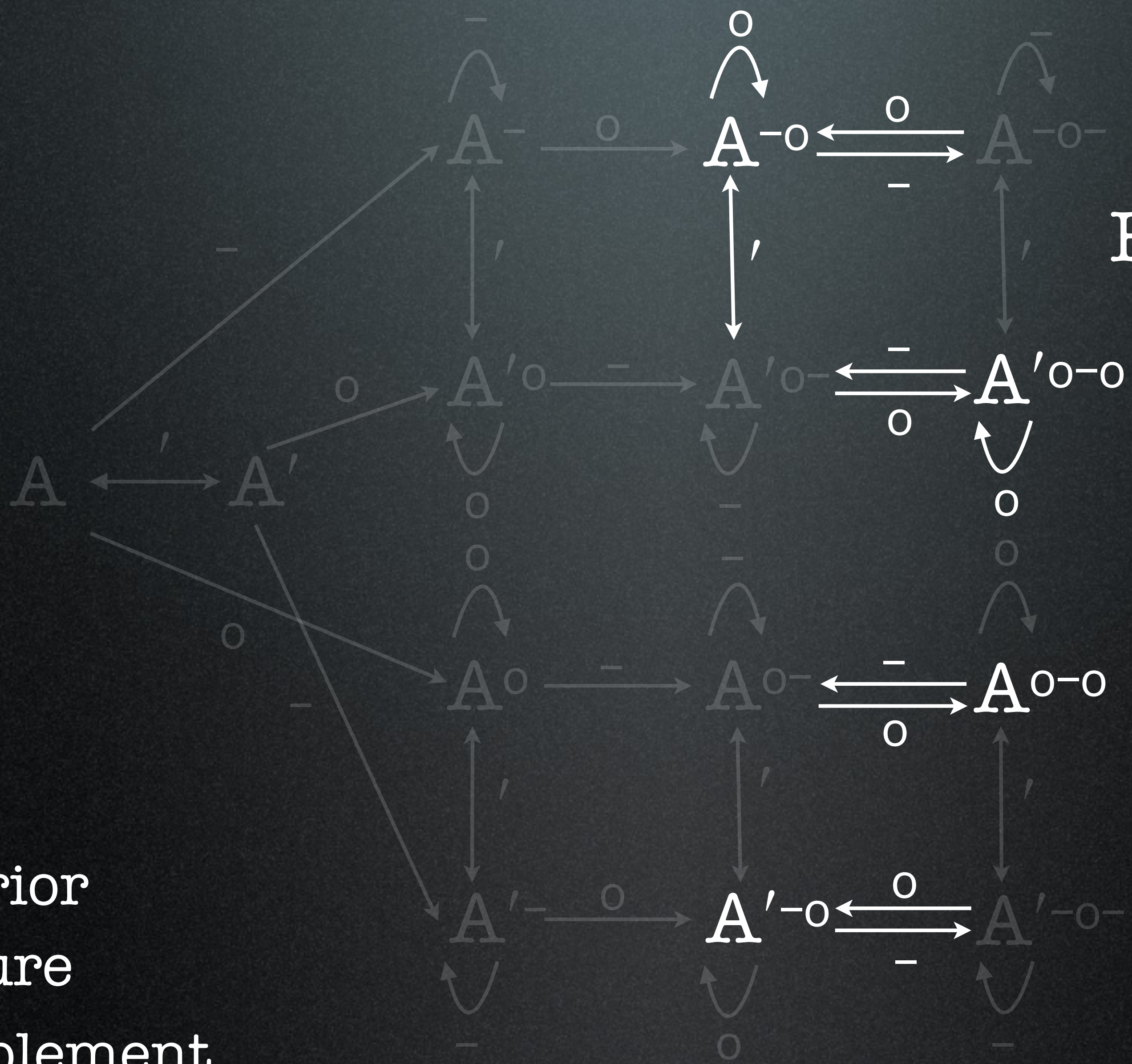
Combinations



Regular open

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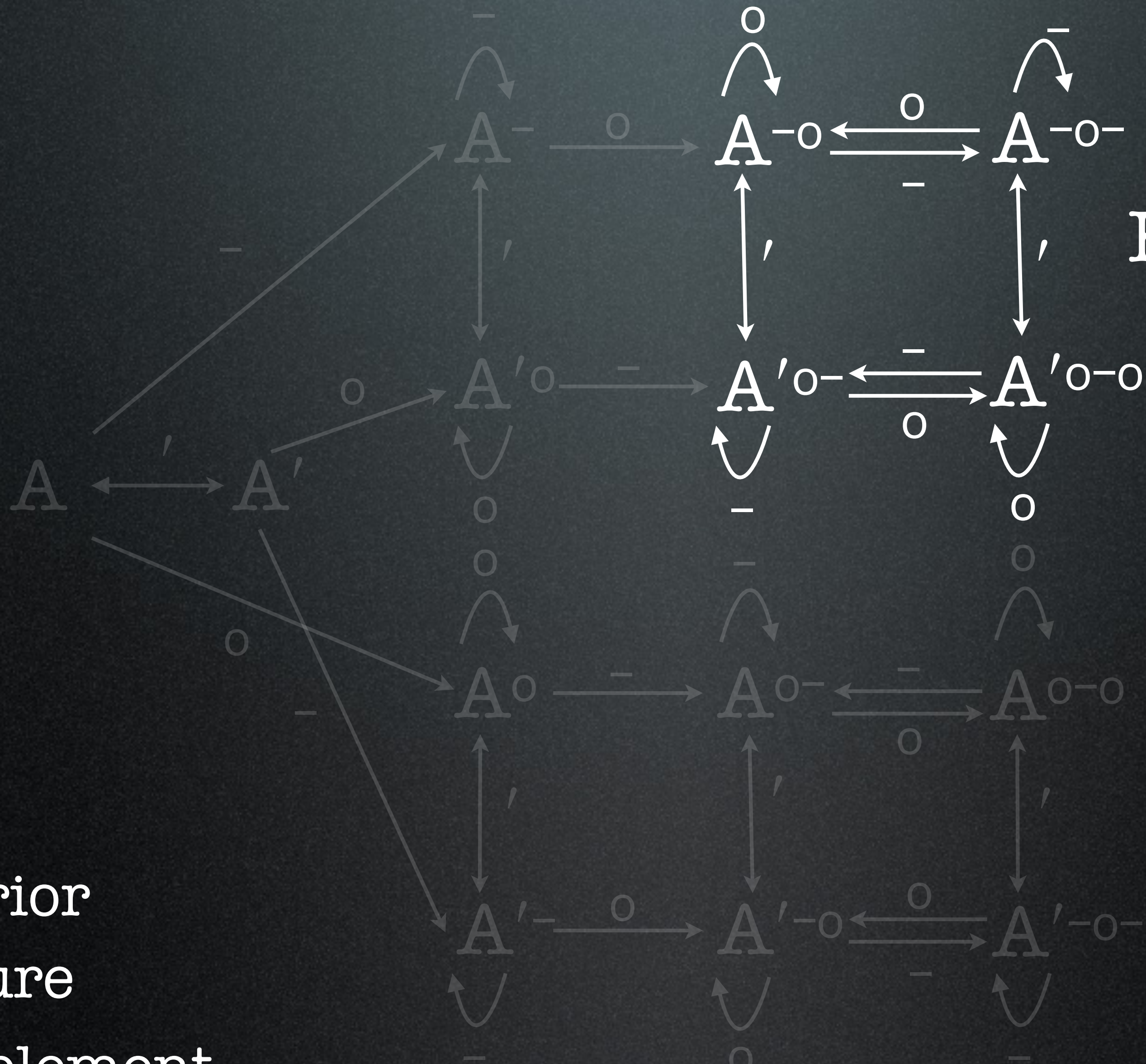
Combinations



Regular open

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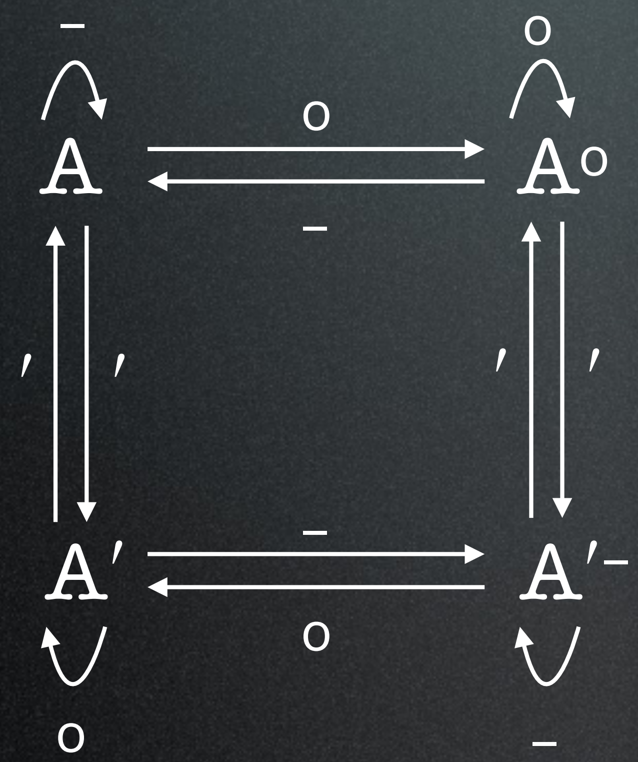
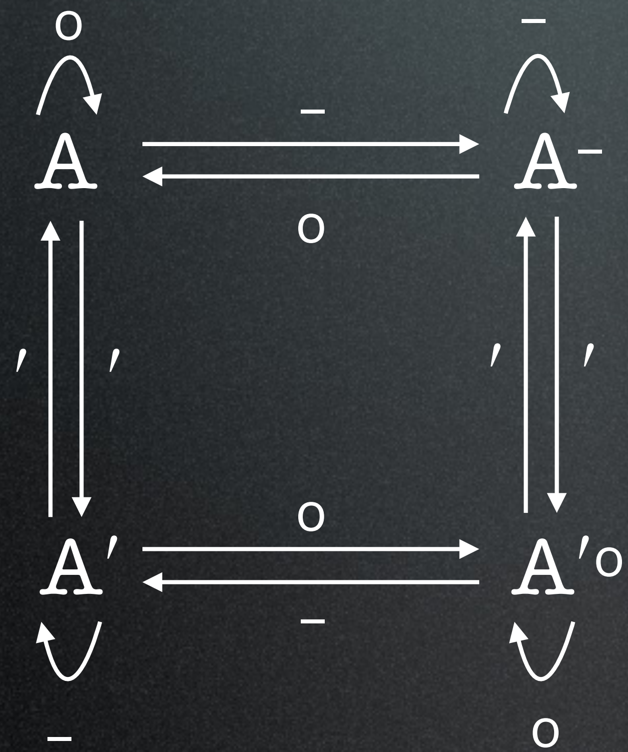
Regular closed regions



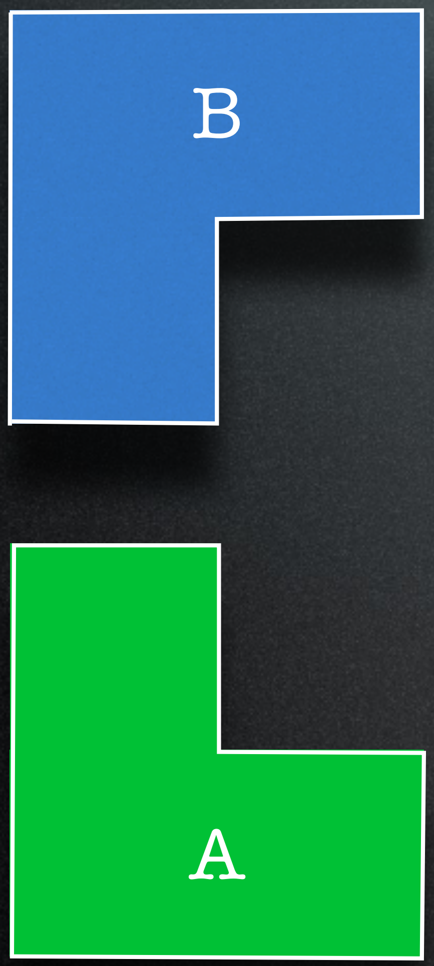
- A regular closed $=_{def}$ $A = A^{\circ -}$
- $-$: closure
- $\circ$: interior
- $'$: complement

Regular open regions



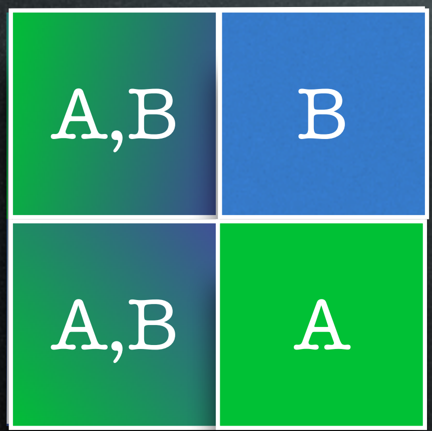
- A regular open $=_{def}$ $A = A^{-o}$
- – : closure
- o : interior
- ′ : complement

Intersection



- A is the union of the closed rectangles [0,0]-[1-2] and [0-1][1-2]
- B is the union of the closed rectangles [0-0]-[1-1] and [0-1]-[2-2]

Intersection



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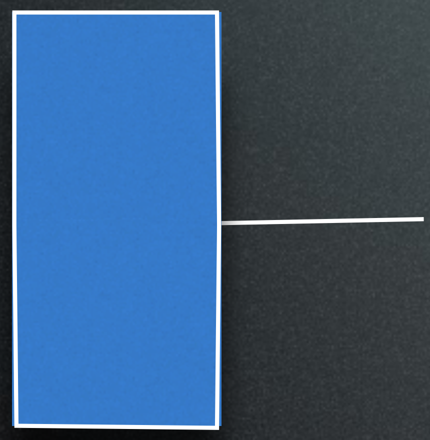
Intersection

$A \cap B$



- The intersection of two regular closed sets (though closed) is not necessarily <u>regular</u> closed.
- Therefore we define the intersection of two regular closed sets as $(A \cap B)^{o-}$

Intersection

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Intersection

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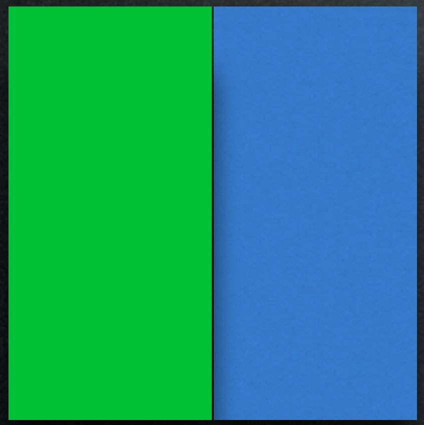
Union

A∪B

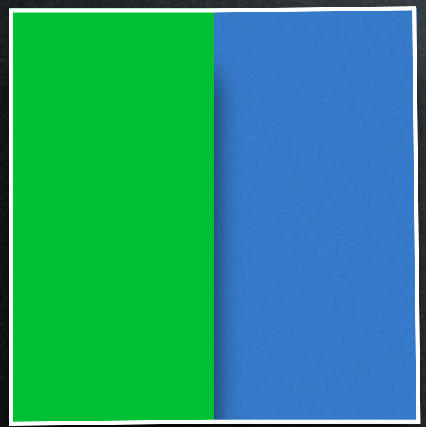


A∪B = <0,0>-<1,2> ∪ <1,0>-<2,2>

- Likewise, the union of two regular open sets (though open) is not necessarily regular open.
- Therefore we define the union of two regular open sets as $(A \cup B)^{-o}$

Union

$(A \cup B)^-$



$(A \cup B)^- = [0,0]-[2,2]$

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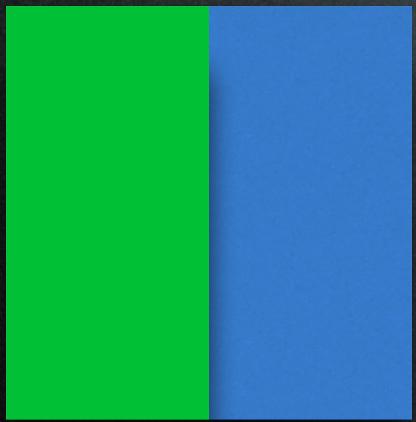
Union

$(A\cup B)^{-o}$



$(A\cup B)^{-o} = \langle 0,0\rangle - \langle 2,2\rangle$

- Likewise, the union of two regular open sets (though open) is not necessarily regular open.
- Therefore we define the union of two regular open sets as $(A\cup B)^{-o}$

Boolean Algebra

- $(A \wedge B) \wedge C = A \wedge (B \wedge C)$
- $(A \vee B) \vee C = A \vee (B \vee C)$
- $A \wedge B = B \wedge A$
- $A \vee B = B \vee A$
- $A \vee (A \wedge B) = A$
- $A \wedge (A \vee B) = A$
- $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$
- $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$
- $A \vee \neg A = \top$
- $A \wedge \neg A = \bot$

Boolean Algebra

- $(A \wedge B) \wedge C = A \wedge (B \wedge C)$
- $(A \vee B) \vee C = A \vee (B \vee C)$
- $A \wedge B = B \wedge A$
- $A \vee B = B \vee A$
- $A \vee (A \wedge B) = A$
- $A \wedge (A \vee B) = A$
- $A \vee (B \wedge C) =$ $(A \vee B) \wedge (A \vee C)$
- $A \wedge (B \vee C) =$ $(A \wedge B) \vee (A \wedge C)$
- $A \vee \neg A = \top$
- $A \wedge \neg A = \bot$

Boolean algebras

This is very classic. It is basically the (even more prototypical) superset algebra with a slight tweak to the complement and union relations (complement and union for the regular closed sets) to make sure the resulting set is a regular open set (resp. a regular closed set)

Regular Open Sets

- $\lfloor \top \rfloor = X$
- $\lfloor \bot \rfloor = \varnothing$
- $\lfloor p \rfloor = p$
- $\lfloor \neg A \rfloor = X \backslash \lfloor A \rfloor^{-}$
- $\lfloor A \wedge B \rfloor = \lfloor A \rfloor \cap \lfloor B \rfloor$
- $\lfloor A \vee B \rfloor = (\lfloor A \rfloor \cup \lfloor B \rfloor)^{-o}$

Regular C

- $\lceil \top \rceil = X$
- $\lceil \bot \rceil = \varnothing$
- $\lceil p \rceil = p$
- $\lceil \neg A \rceil = X \backslash \lceil A \rceil^{o}$
- $\lceil A \wedge B \rceil = (\lceil A \rceil \cap \lceil B \rceil)^{o-}$
- $\lceil A \vee B \rceil = \lceil A \rceil \cup \lceil B \rceil$

Kuratowski Axioms

Closure

- $A \leq A^{-}$
- $A^{--} = A^{-}$
- $\bot^{-} = \bot$
- $(A \cup B)^{-} = A^{-} \cup B^{-}$

S4

- $A \vdash \Diamond A$
- $\Diamond\Diamond A \vdash \Diamond A$
- $\Diamond\bot \vdash \bot$
- $\Diamond(A \vee B) \vdash \Diamond A \vee \Diamond B$
- $\Diamond A \vee \Diamond B \vdash \Diamond(A \vee B)$

Kuratowski Axioms

Interior

- $A^{o} \leq A$
- $A^{o} = A^{oo}$
- $\top = \top^{o}$
- $A^{o} \cap B^{o} = (A \cap B)^{o}$

S4

- $\triangle A \vdash A$
- $\triangle A \vdash \triangle\triangle A$
- $\top \vdash \triangle \top$
- $\triangle A \wedge \triangle B \vdash \triangle (A \wedge B)$
- $\triangle (A \wedge B) \vdash \triangle A \wedge \triangle B$

Kuratowski Axioms

Interior

- $A^{o} \leq A$
- $A^{o} = A^{oo}$
- $\top = \top^{o}$
- $A^{o} \cap B^{o} = (A \cap B)^{o}$

S4

- $!A \vdash A$
- $!A \vdash !!A$
- $\top \vdash !\top$
- $!A \wedge !B \vdash !(A \wedge B)$
- $!(A \wedge B) \vdash !A \wedge !B$

Of course, the logical rules for the exponentials of linear logic are just the rules for S4 as well!

Closure/Interior algebras

Interior Algebra

- $A^{o} \leq A$
- $A^{oo} = A^{o}$
- $\top^{o} = \top$
- $(A \cap B)^{o} = A^{o} \cap B^{o}$

Closure Algebra

- $A \leq A^{-}$
- $A^{--} = A^{-}$
- $\bot^{-} = \bot$
- $(A \cup B)^{-} = A^{-} \cup B^{-}$

Modeling

- Let's turn to some possible applications.
- How would we model a statement like "the interior of region A and the interior of region B have a non-empty intersection"?
- First try: $\neg(\triangle A \wedge \triangle B \leftrightarrow \bot)$

Modeling

- First try: $\neg(\triangle A \triangle B \leftrightarrow \bot)$
- This formula is equivalent to $\triangle A \triangle B$
- Let's construct a model of this formula.

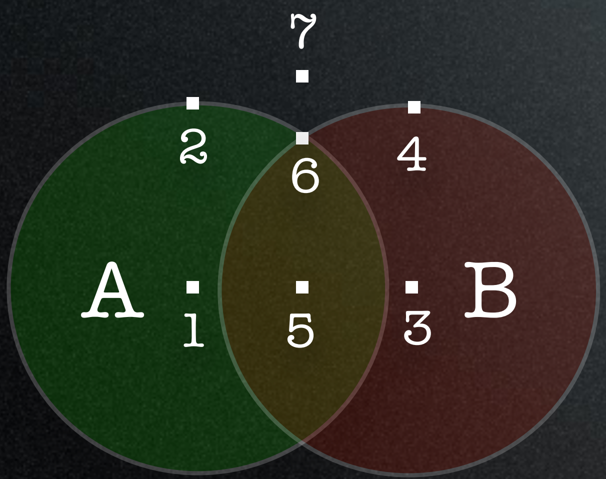
A model of $A^{o} \wedge B^{o}$

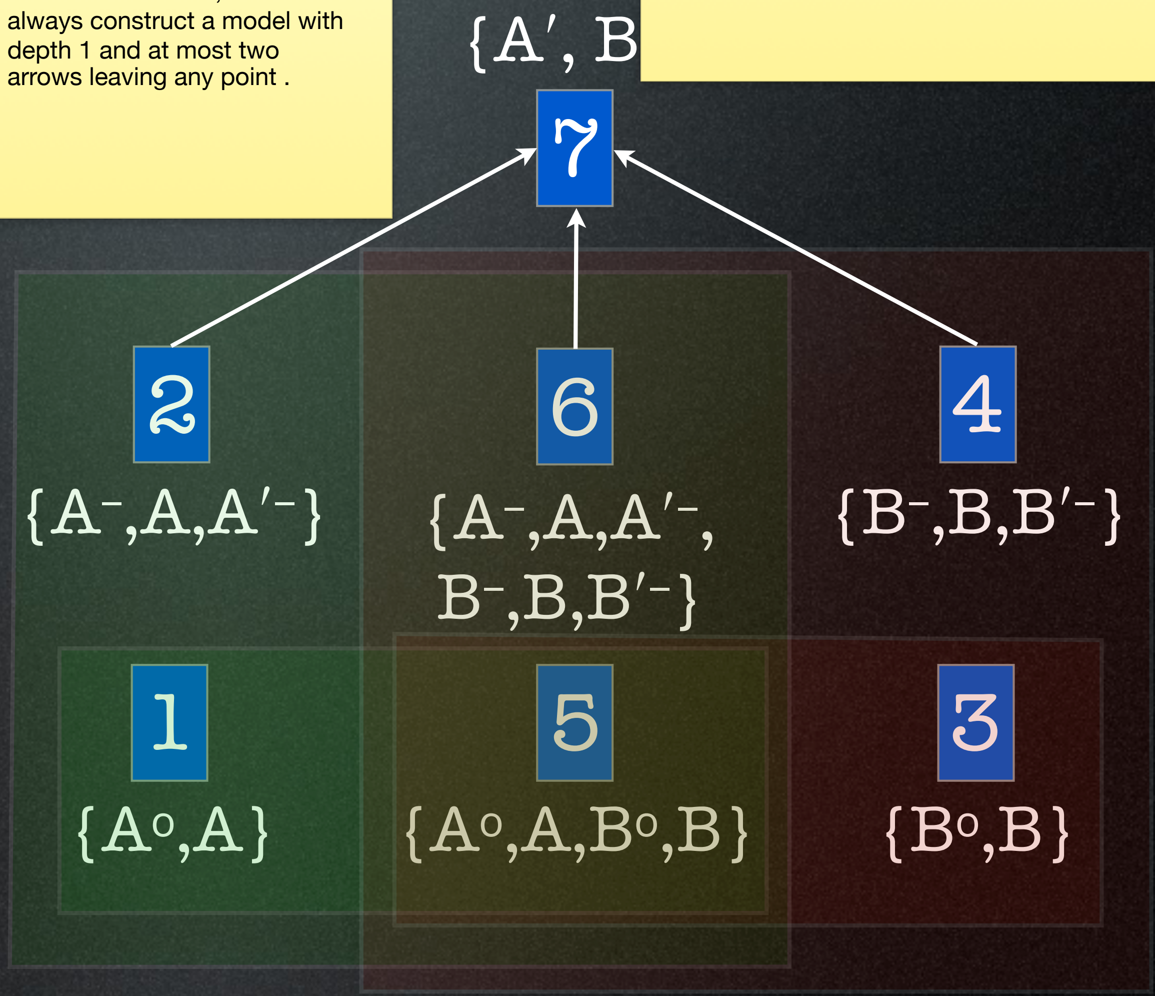
Remark: none of
s are d

Of course, this is just one of many possible models!
In particular, this is a model where A union B is not equal to the universe

The model seems very simple. This is no accident, we can always construct a model with depth 1 and at most two arrows leaving any point .

Question: can we add arrow from 1 to 2?
Answer: No! If we add an arrow from 1 to 2, transitivity would force us to add an arrow from 1 to 7 as well and then A would no longer hold at all states accessible from 1.



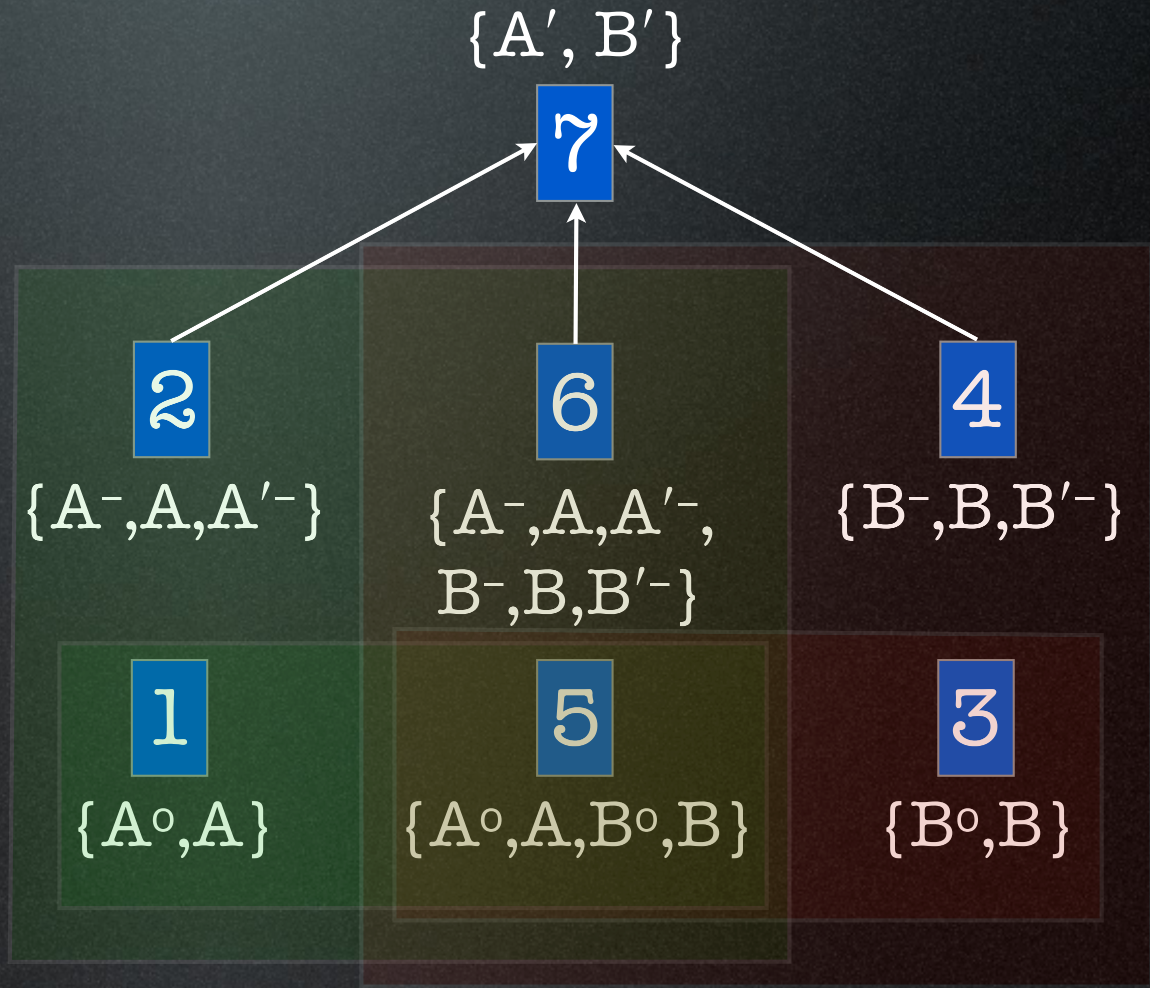


Reminder: A- means there is an arrow to a state where A holds.
Ao means all arrows lead to a state where A holds.

A model of $A^{o} \wedge B^{o}$

Now, in order for the formula $A^{o} \wedge B^{o}$ to hold, it has to be true at every point.

This means that for every point and every point we can reach from this point by following an arrow both A and B must hold.



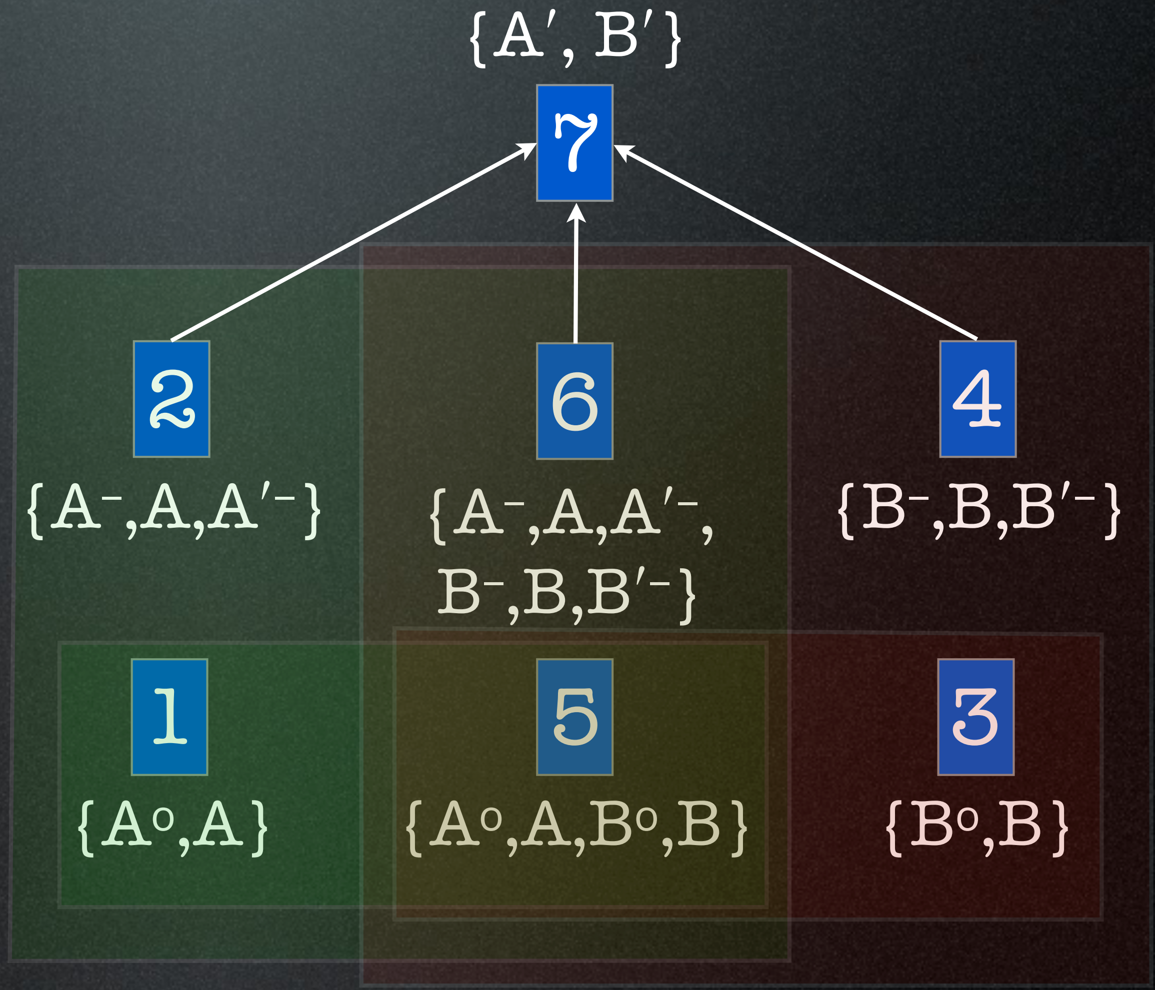
Let’s look at point 7.

A model of $A^{\circ} \wedge B^{\circ}$

The only point we can reach from point 7 is point 7 itself.

Neither A nor B holds at point 7.

Therefore, this model is a countermodel to $A^{\circ} \wedge B^{\circ}$

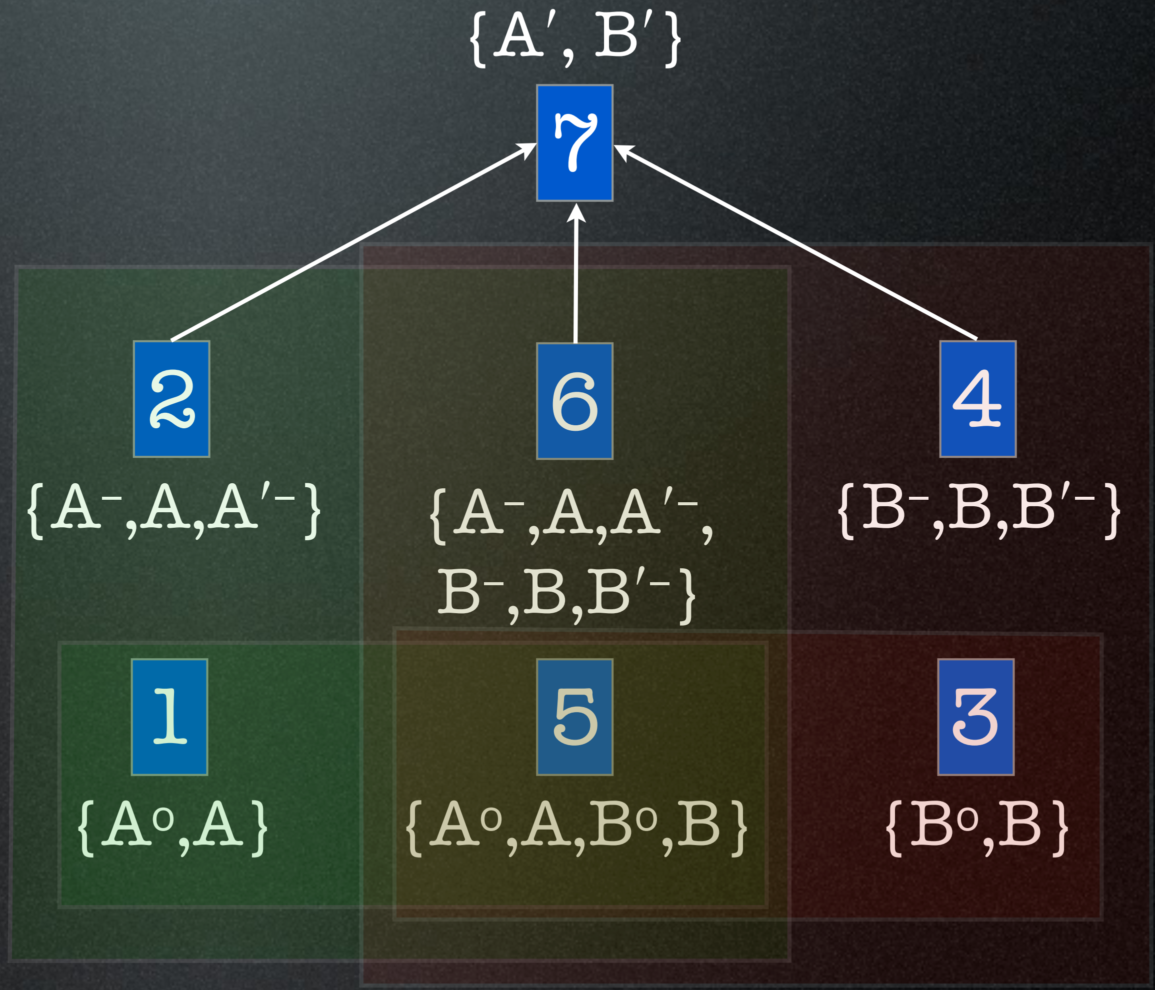


A model of $A^{o} \wedge B^{o}$

We want to state that $A^{o} \wedge B^{o}$ is true at at least one point in the model.

This would correctly model the fact that this intersection is not empty

However, in standard S4 we cannot express this.



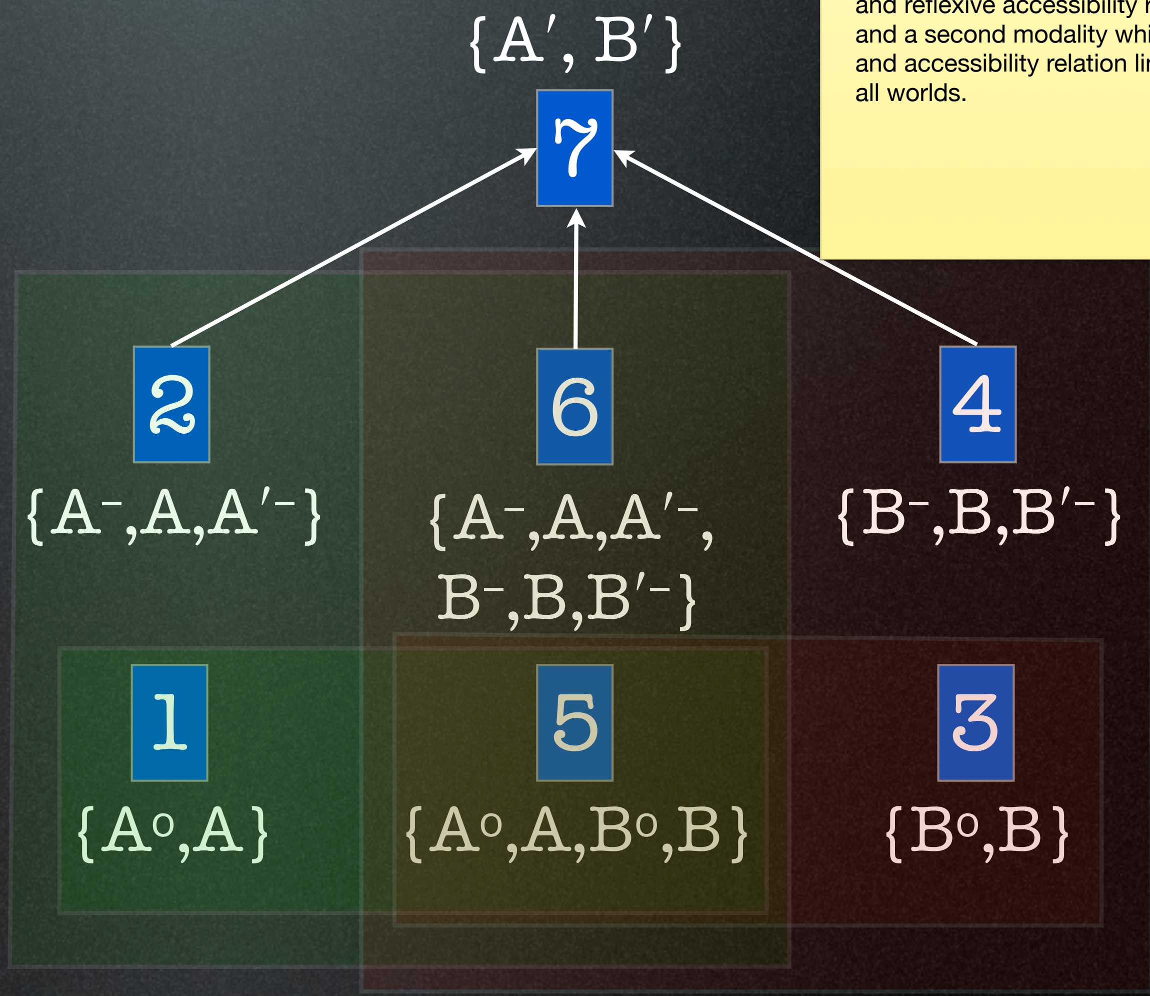
A model of $A^{\circ} \wedge B^{\circ}$

A solution is to add a universal modality to S4, giving the system $S4_u$

One way to see the system S4u is that it is a multimodal system with one S4 modality (with a transitive and reflexive accessibility relation) and a second modality which has and accessibility relation linking all worlds.

A formula ∀F is true if for all points in the model the formula F is true.

A formula ∃F is true is F holds at at least one point.

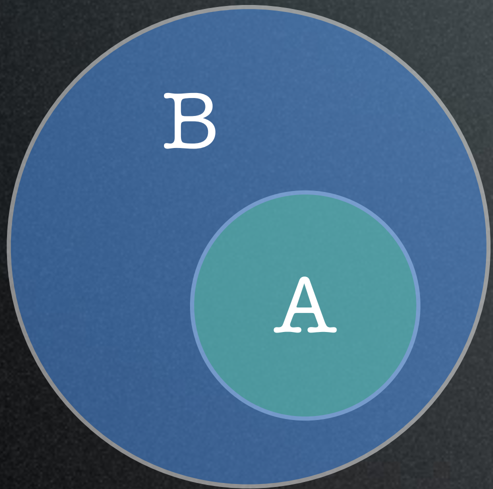


$S4_u$

- In other words, $\forall F$ will mean $|F| = X$ and $\exists F$ will mean $|F| \neq \emptyset$
- The negated forms are interpreted as expected: $\neg\forall F$ will mean $|F| \neq X$ and $\neg\exists F$ will mean $|F| = \emptyset$
- In the following, I will often use formulas containing "$F \neq \emptyset$", "$F \neq$ "$F = \emptyset$", "$F = X$".

This is a slight abuse of notation, but, in my opinion, it makes the formulas a lot easier to read!

$S4_u$ expressivity



- Define $A \subseteq B$ as $\forall(\neg A \vee B)$ or $\neg A \vee B = X$ or $A \rightarrow B = X$
- Define $A \not\subseteq B$ as $\neg\forall(\neg A \vee B)$ or $\neg A \vee B \neq X$ or $A \rightarrow B \neq X$
- Define $A \subset B$ as $A \subseteq B \wedge B \not\subseteq A$

$S4_u$ expressivity

$A \wedge B = \emptyset$

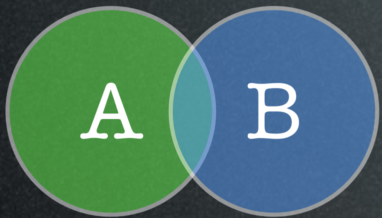
A ->B	B->A	
0	0	DC,EC,PO
0	1	TPP-1 NTPP-1
1	0	TPP NTPP
1	1	EQ

$A^o \wedge B^o = \emptyset \wedge A \wedge B \neq \emptyset$

- Define DC(A,B) as $\neg\exists(A \wedge B)$, which is equivalent to $A \wedge B = \emptyset$
- Define EC(A,B) as $\neg\exists(A^o \wedge B^o) \wedge \exists(A \wedge B)$, or $A^o \wedge B^o = \emptyset \wedge A \wedge B \neq \emptyset$

$S4_u$ expressivity

The interiors share a point but neither A->B not B->A



$$A^{\circ} \wedge B^{\circ} \neq \emptyset \ \wedge \ A^{\circ} \wedge \neg B \neq \emptyset \ \wedge \ \neg A \wedge B^{\circ} \neq \emptyset$$

$$A^{\circ} \wedge B^{\circ} \neq \emptyset \ \wedge \ A \rightarrow B \neq X \ \wedge \ B \rightarrow A \neq X$$

$$A^{\circ} \wedge B^{\circ} \neq \emptyset \wedge A \not\subseteq B \wedge B \not\subseteq A$$

Both A ->B and B->A. The remaining cases are therefore A->B and not B->A (-B/\A) and B->A and not A->B (-A/\B)

$$A \leftrightarrow B = X$$

$$A \subseteq B \wedge B \subseteq A$$

Many authors use the first version (interiors intersected with negations), some others use the second version. Look at the differences and try to find if these are important. The first version intersects only closed sets, whereas the second intersects and open and a closed set, which seems to be an advantage.

- Define PO(A,B) as $\exists(A^{\circ} \wedge B^{\circ}) \wedge \neg(A \subseteq B) \wedge \neg(B \subseteq A)$ or $A^{\circ} \wedge B^{\circ} \neq \emptyset \wedge \neg(A \subseteq B) \wedge \neg(B \subseteq A)$

Define EQ(A,B) as $A \subseteq B \wedge B \subseteq A$.

$S4_u$ expressivity

Can we conclude from A->Bo that ¬(B->A)? Does this require Bo to be a proper subset of B (which is not required)? (check the name of a topological space where the only clopen sets are the empty set and the universe)
I

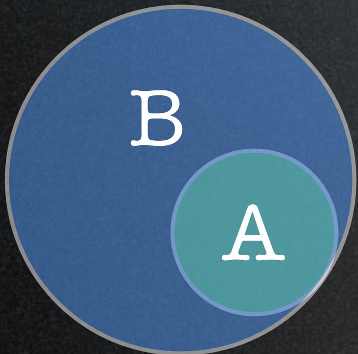
ne
P(A,B) as
$B^o = X \wedge$
$\neg A \wedge B \neq \emptyset$

$\neg A \vee B^o = X \wedge \neg A \wedge B \neq \emptyset$

$A \subseteq B^o \wedge B \not\subseteq A$

- Define TPP(A,B) as $\neg A \vee B = X \wedge$

A ->B and ¬(B->A) and ¬(A->Bo)



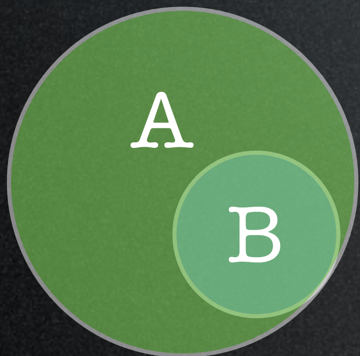
$\neg A \vee B = X \wedge \neg A \wedge B \neq \emptyset \wedge A \wedge \neg(B^o) \neq \emptyset$

$A \subseteq B \wedge A \not\subseteq B^o \wedge B \not\subseteq A$

$S4_u$ expressivity

$\neg B \vee A^{o} = X \wedge \neg B \wedge A \neq \emptyset$

$B \subseteq A^{o} \wedge A \not\subseteq B$



$\neg B \vee A = X \wedge \neg B \wedge A \neq \emptyset \wedge B \wedge \neg(A^{o}) \neq \emptyset$

$B \subset A \wedge B \not\subseteq A^{o}$

- Define $NTPP^{-1}(A,B)$ as $NTPP(B,A)$
- Define $TPP^{-1}(A,B)$ as $TPP(B,A)$

$S4_u$ expressivity

- We have shown that there are formulas defining the RCC8 relations in $S4_u$
- A natural question is: are there any useful things we can express in $S4_u$ which are not expressible in RCC8
- Since the RCC8 relations apply only to region variables, it seems natural to consider complex formulas built from region variables.

$S4_u$ expressivity

- Since the RCC8 relations apply only to region variables, it seems natural to consider complex formulas built from region variables.
- The resulting calculus is sometimes called BRCC8.

S4u expressivity

- EQ(UnionEuropéene, PaysBays∨Belgique∨France∨...)
- EQ(Aquitaine, Dordogne ∨ Gironde ∨ Landes ∨ LotEtGaronne ∨ PyrénéesAtlantiques)
- TPP(Pyrénées, France∨Espagne∨A
- EC(France∧Pyrénées,Espagne∧Pyr

We can use equality statements to specify that a region is exactly the union of a number of other regions. In RCC8 we can only specify that each of the different regions is a part of (tangential or not) a super-region but not the inverse.

These two statements capture the fact that France and Spain are connected by means of the Pyrénées. This a stronger than the RCC8 statements PO(France,Pyrénées) PO(Espagne,Pyrénées) EC(France, Espagne)

S4u expressivity

- EC(Andorre,France∧Pyrénées)
- EC(Andorre,Espagne∧Pyrénées)
- NTPP(Andorre,Pyrénées)
- EQ(France, FranceContinental∨Corse)
- DC(FranceContinental,Corse)

We can state that a region denoted by a certain variable is discontinuous.
This is impossible in RCC8

S4u expressivity

- TPP(Pyrénées, France∨Espagne∨Andorre)
- EC(France∧Pyrénées,Espagne∧Pyrénées)
- DC(France∧¬Pyrénées,Espagne∧¬
- EC(Andorre,France∧Pyrénées)
- EC(Andorre,Espagne∧Pyrénées)
- NTPP(Andorre,Pyrénées)

These two statements capture the fact that France and Spain are connected by means of the Pyrénées. This a stronger than the RCC8 statements PO(France,Pyrénées) PO(Espagne,Pyrénées) EC(France, Espagne)